Topological Relationships among Folded hypercubes, Even graphs and Odd graphs

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Abstract - *In this paper, we establish topological relationships among Folded hypercube FQ_n, Even graph E_k and Odd graph O_d via embedding. FQ_n can be embedded into E_{n+1} and O_{n+1} with dilation 2 and congestion 1. And E_k can be embedded into FQ_{2k-3} with dilation 1 and into O_k with dilation 2 and congestion 1. Also O_d can be embedded into E_{d+1} and FQ_{2d-1} with dilation 2 and congestion 1.*

Keywords : Interconnection network, Folded hypercube, Even graph, Odd graph, Embedding

1 Introduction

Study of the properties of an interconnection network is an important part of the study of any parallel processing or distributed system. The hypercube is widely used and well-known interconnection model since it possesses many attractive properties - low diameter, relatively small degree, recursive structure, etc. One of the biggest reasons for the popularity of the hypercube is its ability to efficiently embed many parallel architectures[1, 3, 8, 10, 12]. Consequently, Several attempts have been made to generalize and specialize hypercubes. For instance, Folded hypercube[4], Even network[5], Odd graph[2]. In this paper, we establish embedding relationships among these three networks with dilation and congestion.

In interconnection networks, the problem of simulating one network by another is modelled as a graph embedding problem. Let G be a graph, its node set, edge set and set of path in G will be denoted by $V(G)E(G)$ and $P(G)$, respectively. An embedding (Ψ,ρ) of $G(V,E)$ into a graph $G'(V',E')$ is an injective mapping of nodes of $V(G)$ into the nodes of $V'(G')$ and edges of $E(G)$ into the paths of $P(G')$, i.e., Ψ: $V \rightarrow V'$ and ρ: $E \rightarrow P(G')$. We refer to G as the guest graph and G' as the host graph. Important measures of quality of an embedding are dilation and congestion. The dilation is the length of the longest path of G' corresponding to an edge of G, the congestion is the largest number of edges of G whose images contain the same edge of G'. There are several reasons why such an embedding is important[13]. Graph embedding results have many important applications in parallel processing. They provide the theoretical foundation for studying the problem of matching the communication structure of a task to the communication support of a parallel system and, also, for studying the problem of evaluating the relative performance of two interconnection networks. Most of them consider the embedding in which the size of the guest network is equal to or less than the size of the host network.

In the next section, we introduce the definition of Folded hypercube FQ_n, Even graph E_k and Odd graph O_d. In section 3,we show embedding results, and we present a conclusion.

2 Preliminaries

A binary string x of length n will be written as $x_1x_2 \cdots x_i \cdots x_n$, where $x_i \in \{0,1\}, 1 \leq i \leq n, x_i$ is said to be the ith bit, the complement of x_i will be denoted by $\overline{x_i} = 1 - xi$. Let s and t be two binary strings. The Hamming distance between s and t is the number of bits that they differ.
The n-dimensional folded hypercube FQ_n proposed first by El-Amawy and Latifi[4]. The number of nodes in FQ_n is $2n$, degree of FQ_n is n + 1 and each node with a distinct binary string is $x_1x_2 \cdots x_n$. Two nodes in FQ_n are adjacent if and only if Hamming distance is 1 or n. An edge connecting two nodes u and v is denoted i-edge, where Hamming distance is 1. And an edge connecting two nodes a and b is denoted c-edge, where Hamming distance is n. It has diameter $\frac{n}{2}$, about half the diameter of hypercube[4], and FQn is a bipartite graph if and only if n is odd[14]. The study of the folded hypercube has recently attracted the attention of many researchers[9, 11, 14]. Fig. 1 shows a Folded hypercube FQ_3.
In [5], Ghafoor introduced even graph E_k by a class of fault-tolerant multiprocessor networks. It was analyzed some important properties such as maximal fault-tolerance, node and edge symmetry, node disjoint paths, routing algorithms during fault-free and faulty conditions, and ease of self-diagnosis[5, 6]. The number of nodes in E_k is $\binom{2k-2}{k-1}$, degree of E_k is k and its diameter is k - 1. Each node with a distinct binary string is $x_1x_2 \cdots x_{2k-3}(|0| = |1| \pm 1)$.$|t|$ is the number of t. Two nodes in E_k are adjacent if and only if Hamming

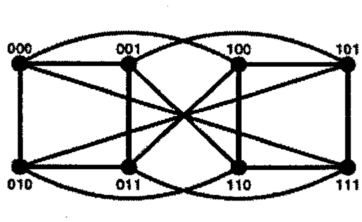

Fig. 1. FQ_3

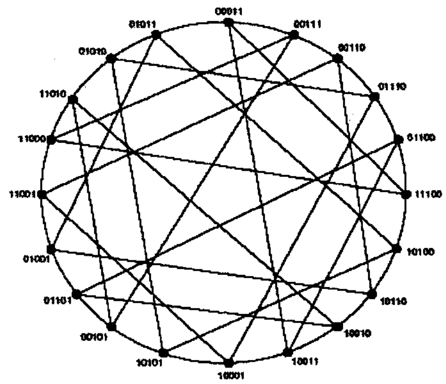

Fig. 2. E_4

distance is 1 or $2k$ - 3. An edge connecting two nodes u and v is denoted i-edge, where Hamming distance is 1. And an edge connecting two nodes a and b is denoted c-edge, where Hamming distance is $2k$ - 3. Fig. 2 shows a Even graph E_4.
The class of Odd graphs was introduced by [2] in the context of graph theory. [7] pointed out their potential as interconnection networks and they discussed various properties. Odd graph O_d with $d \geq 2$ has the set of binary strings of length $2d$ - 1 with exactly d 1's as the node set. The number of nodes in O_d is $\binom{2d-1}{d}$, degree of O_d is d and its diameter is d - 1. Two nodes are adjacent if and only if they differ in all but one position; in other words, two nodes are adjacent if and only if their Hamming distance is $2d$ - 2. An edge connecting two nodes u and v is denoted i-edge, where Hamming distance is $2d$-2 Fig. 3 shows a Odd graph O_4.

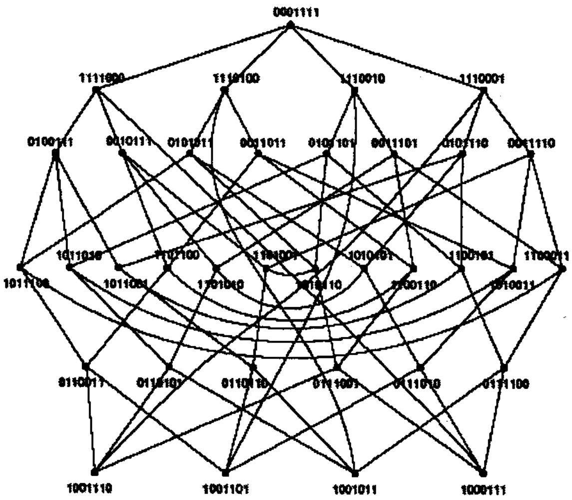

Fig. 3. O_4

3 Embedding

Let V_n^1 and V_n^0 be the set of nodes that start with 1 and 0. Let S_n^1 be the set of nodes that $|1| = |0|+1$ and S_n^0 be the set of nodes that $|0| = |1|+1$.

Theorem 1. Even graph E_k is a bipartite graph.

Proof. Let s_1 and s_2 be two binary strings and $u= s_1 0 s_2 (0 \le |s_1, s_2| \le 2k-4) \in S_k^0$ in E_k. Then u is connected $u' = s_1 1 s_2$ or $u''=\overline{s_1} 1 \overline{s_2}$ by the definition of E_k. As the result, we know $u', u'' \in S_k^1$. Hence the proof is complete. □

Theorem 2. Folded hypercube FQ_n can be embedded into Even graph E_{n+1} with dilation 2 and congestion 1.

Proof. Let $u = b_1 b_2 \cdots b_n$ be an arbitrary node in FQ_n and $\overline{u}= \overline{b_1 b_2 \cdots b_n}$ be a complement node of u. We use t to denote the string obtained from u by deleting the leftmost bit. Throughout this proof, we use s_1s_2 to denote the concatenation of the two binary strings s_1 and s_2. For each node $u\in V(FQ_n)$, define $\Psi(u) = t\overline{u}= b_2 \cdots b_n\overline{b_1 b_2 \cdots b_n}$. Then $\Psi(u)$ is a binary string of length $2n$ - 1, $|0| = |1| \pm 1$. Hence Ψ: $V(FQ_n) \rightarrow V(E_{n+1})$ is a mapping. Now consider the edges of FQ_n. Let $e \in E(FQ_n)$. We have three cases.

Case 1. The edge e is c-edge : Let the two adjacent nodes be $u, v= \overline{u}$. Then we may assume they are $\Psi(u) = t\overline{u}$ and $\Psi(v) = \overline{t}u$ where $\Psi(u)$ and $\Psi(v)$ are binary strings of length $2n$ - 1. Define $\rho(e)$ to be the path of length $1: (t\overline{u} - \overline{t}u)$ in E_{n+1}.

Case 2. The two adjacent nodes by i-edge differ in the first position : Let the two adjacent nodes be $u = 0t$; $v = 1t$. Then we may assume they are $\Psi(u) = t1\overline{t}$ and $\Psi(v) = t0\overline{t}$. Define $\rho(e)$ to be the path of length $1: (t1\overline{t} - t0\overline{t})$ in E_{n+1}.

Case 3. The two adjacent nodes by i-edge are the same in the first position : Let the two adjacent nodes be $u= xs_1 0 s_2, v= xs_1 1 s_2$ where x is 0 or 1, and s_1 and s_2 are binary strings$(0 \le |s_1, s_2| < n-1)$. Then we may assume they are $\Psi(u)= s_1 0 s_2 \overline{x}\overline{s_1} 1 \overline{s_2}$ and $\Psi(v) = s_1 1 s_2 \overline{x}\overline{s_1} 0 \overline{s_2}$. Define $\rho(e)$ to be the path of length 2 :

$(s_1 0 s_2 \overline{x}\overline{s}_1 1 \overline{s}_2 - s_1 0 s_2 \overline{x}\overline{s}_1 0 \overline{s}_2 - s_1 1 s_2 \overline{x}\overline{s}_1 0 \overline{s}_2)$

in E_{n+1}.

We know easily that the node $s_1 0 s_2 \overline{x}\overline{s}_1 0 \overline{s}_2$ in case 3 is not same the node $\Psi(u)$ or $\Psi(v)$ in case 1 and 2. Therefore, each path $\rho(e)$ in above three cases is edge-disjoint path. Hence the proof is complete. □

Theorem 3. Folded hypercube FQ_n can be embedded into Odd graph O_{n+1} with dilation 2, congestion 1.

Proof. Let $u = b_1 b_2 \cdots b_n$ be an arbitrary node in FQ_n and $\overline{u}= \overline{b_1 b_2 \cdots b_n}$ be a complement node of u. We use t to denote the string obtained from u by deleting the leftmost bit. Throughout this proof, we use $s_1 s_2 1$ to denote the concatenation of the three binary strings s_1, s_2 and 1. For each node $u\in V(FQ_n)$, define

$\Psi(u) = u\overline{u}1= b_1 b_2 \cdots b_n\overline{b_1 b_2 \cdots b_n}1$. Then $\Psi(u)$ is a binary string of length $2n+1, |1| = |0|+1$. Hence Ψ: $V(FQ_n)\rightarrow V(O_{n+1})$ is a mapping. Now consider the edges of FQ_n. Let $e \in E(FQ_n)$. We have three cases.

Case 1. The edge e is c-edge : Let the two adjacent nodes be $u, v= \overline{u}$. Then we may assume they are $\Psi(u) = u\bar{u}1$ and $\Psi(v) = \bar{u}u1$ where $\Psi(u)$ and $\Psi(v)$ are binary strings of length $2n + 1$. Define $\rho(e)$ to be the path of length $1: (\bar{u}u1 - u\bar{u}1)$ in O_{n+1}.

Case 2. The two adjacent nodes by *i*-edge differ in the first position : Let the two adjacent nodes be $u = 0t$, $v = 1t$. Then we may assume they are $\Psi(u) = 0t1\bar{t}1$ and $\Psi(v) = 1t0\bar{t}1$. Define $\rho(e)$ to be the path of length $1: (0t1\bar{t}1 - 1t0\bar{t}1)$ in O_{n+1}.

Case 3. The two adjacent nodes by *i*-edge are the same in the first position : Let the two adjacent nodes be $u = xs_1 0 s_2, v = xs_1 1 s_2$ where x is 0 or 1, and s_1 and s_2 are binary strings$(0 \leq |s_1, s_2| < n-1)$. Then we may assume they are $\Psi(u) = xs_1 0 s_2 \bar{x}\bar{s}_1 1 \bar{s}_2 1$ and $\Psi(v) = xs_1 1 s_2 \bar{x}\bar{s}_1 0 \bar{s}_2 1$. Define $\rho(e)$ to be the path of length $2: (xs_1 0 s_2 \bar{x}\bar{s}_1 1 \bar{s}_2 1 - \bar{x}\bar{s}_1 1 \bar{s}_2 x s_1 1 s_2 0 - xs_1 1 s_2 \bar{x}\bar{s}_1 0 \bar{s}_2 1)$ in O_{n+1}. We know easily that the node $\bar{x}\bar{s}_1 1 \bar{s}_2 x s_1 1 s_2 0$ in case 3 is not same the node $\Psi(u)$ or $\Psi(v)$ in case 1 and case 2. Therefore, each path $\rho(e)$ in above three cases is edge-disjoint path. Hence the proof is complete. □

Theorem 4. Even graph E_k can be embedded into $(2k-3)$-dimensional Folded hypercube FQ_{2k-3} with dilation 1.

Proof. FQ_{2k-3} is a bipartite graph[14], and E_k is a bipartite graph by theorem 1. Let s_1 and s_2 be two binary strings in $E_k, 0 \leq |s_1, s_2| < k$. Hence Ψ: $V(E_k) \rightarrow V(FQ_{2k-3})$ is a mapping. Now consider the edges of E_k.
Let $e \in E(E_k)$. We have two cases.

Case 1. The edge e is a *i*-edge : Let the two adjacent nodes be $u = s_1 0 s_2$ and $v = s_1 1 s_2$ in E_k. Then we know $\Psi(u) = s_1 0 s_2$, $\Psi(v) = s_1 1 s_2$. Define $\rho(e)$ to be the path of length $1: (s_1 0 s_2 \rightarrow s_1 1 s_2)$ in FQ_{2k-3}.

Case 2. The edge e is a *c*-edge : Let the two adjacent nodes be $a = s_1 0 s_2$ and $b = \bar{s}_1 1 \bar{s}_2$ in E_k. Then we know $\Psi(a) = s_1 0 s_2$, $\Psi(b) = \bar{s}_1 1 \bar{s}_2$. Define $\rho(e)$ to be the path of length $1: (s_1 0 s_2 \rightarrow \bar{s}_1 1 \bar{s}_2)$ in FQ_{2k-3}. Hence the proof is complete. □

Theorem 5. Even graph E_d can be embedded into Odd graph O_d with dilation 2, congestion 1.

Proof. Let $u = b_1 b_2 \cdots b_{2d-3}$ be an arbitrary node in E_d and $\bar{u} = \overline{b_1 b_2 \cdots b_{2d-3}}$ be a complement node of u. If $u \in V_d^1$, then $\Psi(u) = 1\bar{u}0$ and If $v \in V_d^0$, then $\Psi(u) = 0u1$. Then $\Psi(u)$ is a binary string of length $2d - 1$, $|0| = |1| \pm 1$. Hence Ψ: $V(E_d) \rightarrow V(O_d)$ is a mapping. Now consider the edges of E_d. Let $e = E(E_d)$. We have two cases.

Case 1. The edge e is *i*-edge : Let the two adjacent nodes be $u = s_1 1 s_2, v = s_1 0 s_2 (0 \leq |s_1, s_2| \leq 2d-4)$. We know E_d is a bipartite graph by theorem 1, we may assume $u \in V_d^1$ and $v \in V_d^0$. So $\Psi(u) = 0 s_1 1 s_2 1$, $\Psi(v) = 1 \bar{s}_1 1 \bar{s}_2 0$. Define $\rho(e)$ to be the path of length $1: (0 s_1 1 s_2 1 - 1 \bar{s}_1 1 \bar{s}_2 0)$ in O_d.

Case 2. The edge e is *c*-edge : Let the two adjacent nodes be $u = s_1 0 s_2$, $v = \bar{s}_1 1 \bar{s}_2 (0 \leq |s_1, s_2| \leq 2d-4)$. If $u \in V_d^0$ and $v \in V_d^1$, then $\Psi(u) = 1 \bar{s}_1 1 \bar{s}_2 0$, $\Psi(v) = 0 \bar{s}_1 1 \bar{s}_2 1$. Define $\rho(e)$ to be the path of length $2: (1 \bar{s}_1 1 \bar{s}_2 0 - 1 s_1 0 s_2 1 - 0 \bar{s}_1 1 \bar{s}_2 1)$ in O_d. If $u \in V_d^1$ and $v \in V_d^0$, then $\Psi(u) = 0 s_1 0 s_2 1$, $\Psi(v) = 1 s_1 0 s_2 0$. Define $\rho(e)$ to be the path of length $2: (0 s_1 0 s_2 1 - 1 \bar{s}_1 1 \bar{s}_2 1 - 1 s_1 0 s_2 0)$ in O_d. We know easily that two nodes $1 s_1 0 s_2 1$ and $1 \bar{s}_1 1 \bar{s}_2 1$ in case 2 are not same the node $\Psi(u)$ or $\Psi(v)$ in case 1. Therefore, each path $\rho(e)$ in above two cases is edge-disjoint path. Hence the proof is complete. □

Theorem 6. Odd graph O_d can be embedded into Even graph E_{d+1} with dilation 2, congestion 1.

Proof. Let $u = b_1 b_2 \cdots b_{2d-1}$ be an arbitrary node in O_d and $\bar{u} = \overline{b_1 b_2 \cdots b_{2d-1}}$ be a complement node of u. We use t to denote the string obtained from u by deleting the leftmost bit. If $u \in S_d^0$, then $\Psi(u) = 0t$ and If $u \in S_d^1$, then $\Psi(u) = 0\bar{t}$. Then $\Psi(u)$ is a binary string of length $2d - 1$, $|0| = |1| \pm 1$. Hence Ψ: $V(O_d) \rightarrow V(E_{d+1})$ is a mapping. Now consider the edges of O_d. Let $e \in E(O_d)$. We have two cases.

Case 1. The edge e is i-edge$(2 \leq i \leq 2d-1)$: Let the two adjacent nodes be $u = 0s_1 1s_2$, $v = 1\bar{s}_1 1\bar{s}_2 (0 \leq |s1, s2| \leq 2d-3)$. Then we may assume $u \in S_d^0$ and $v \in S_d^1$, so $\Psi(u) = 0s_1 1s_2$, $\Psi(v) = 0s_1 0s_2$. Define $\rho(e)$ to be the path of length $1 : (0s_1 1s_2 - 0s_1 0s_2)$ in E_{d+1}.

Case 2. The edge e is $i-edge(i=1)$: Let $u = 1t, v = 1\bar{t}$. Then $\Psi(u) = 0\bar{t}, \Psi(v) = 0t$. Define $\rho(e)$ to be the path of length $2 : (0\bar{t} - 1t - 0t)$ in E_{d+1}.

We know easily that $1t$ in case 2 is not same the node $\Psi(u)$ or $\Psi(v)$ in case 1. Therefore, each path $\rho(e)$ in above two cases is edge-disjoint path. Hence the proof is complete. □

Theorem 7. Odd graph O_d can be embedded into Folded hypercube FQ_{2d-1} with dilation 2 and congestion 1.

Proof. It is obvious by theorem 4 and theorem 6. □

4 Conclusion

In this paper, we have investigated embeddings among Folded hypercubes, Even graphs and Odd graphs. We proved FQ_n can be embedded into E_{n+1} and O_{n+1} with dilation 2 and congestion 1 and E_k can be embedded into FQ_{2k-3} with dilation 1 and into O_k with dilation 2 and congestion 1, also O_d can be embedded into E_{d+1} and FQ_{2d-1} with dilation 2 and congestion 1. By studying the topological relationships among them, we established the aptness of these graphs as interconnection networks.

5 References

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