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Topological properties of folded hyper-star networks

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Abstract In practice it is important to construct node-disjoint paths in networks, because they can be used to increase the transmission rate and enhance the transmission reliability. The folded hyper-star networks $FHS(2n, n)$ were introduced to be a competitive model to both hypercubes and star graphs. They are bipartite and node-symmetric, though not edge-symmetric, and have diameter n. In this paper we construct a maximum number of node-disjoint paths between every two distinct nodes of $FHS(2n, n)$ and show that its fault diameter is $n + 2$ for $n \geq 4$. We also suggest a one-to-all broadcasting algorithm of $FHS(2n, n)$ under the all-port model.

Keywords Interconnection network · Folded hyper-star · Node-disjoint path · Fault diameter · Bipartite network · Node-symmetry

1 Introduction

Advances in hardware technology, especially in VLSI circuit technology, have made it possible to build a large-scale multiprocessor system that contains thousands or even tens of thousands of processors. One crucial step in designing a large-scale multiprocessor system is to determine the topology of the interconnection network (network for short), because the system performance is significantly affected by the network topology. In recent decades, a number of networks were proposed in the literature [12, 18]. A network is conveniently represented by a graph whose nodes

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represent the processors of the network and whose edges represent the communication links of the network. Throughout this paper, we use network and graph, processor and node, and link and edge, interchangeably.

Let $G = (V, E)$ be a connected graph, where V and E represent the node set and edge set of G, respectively. The *degree* of a node in G is the number of edges incident with it. If all nodes have the same degree d, then G is called *regular*. The *distance* between two nodes u and v, denoted by $\mathrm{dist}(u, v)$, is the length of a shortest path between u and v. The *diameter* of G is the maximum distance between any two nodes of G. The *node-connectivity* of G is the minimum number of nodes in G whose removal causes G to become disconnected or trivial. G is said to be *node-symmetric* if for any two nodes u and v, there exists an automorphism of the graph G that maps u into v. In other words, G has the same shape when viewed from any node. The definition is similar for edge-symmetric. The *fault diameter* of G is the maximum diameter resulting from the deletion of any set of fewer nodes than the node-connectivity of G.

One of the most efficient interconnection networks is the hypercube [13]. Another family of regular graphs, the star graphs [1], has been extensively studied. As an application, in [2, 3] reconfigurable interconnection networks were introduced that provide all good properties of the hypercube and at the same time offer more scalability. The hyper-star graphs $HS(m, n)$ were introduced by Lee et al. [17] and Kim et al. [14] to become a new type of interconnection networks competing with both hypercubes and star graphs. The hyper-star network is a regular network when $m = 2n$. Inspired by the idea of El-Amawy and Latifi [9] who proposed the so-called *folded hypercubes* to strengthen the structure of hypercubes, a variation of hyper-star graphs was introduced in [17] as follows: The *folded hyper-star* $FHS(2n, n)$ is constructed from $HS(2n, n)$ by adding edges to connect pairs of nodes whose binary strings are complement to each other (and thus having largest distance in the hyper-star). A result in [17] also showed that hyper-star and folded hyper-star graphs gave lower network cost (measured by the product of degree and diameter) than hypercubes, folded hypercubes, and other variants. Later on, more strong structural properties and some embedding schemes for hypercubes and hyper-star graphs were provided in [5, 6] and [15, 25], respectively.

In practice it is important to construct node-disjoint paths in networks, because they can be used to increase the transmission rate and enhance the transmission reliability. Besides that, node-disjoint paths have applications in multipath routing (such as Rabin's information dispersal algorithm [20], fault tolerance [8, 10, 22], distance distribution of nodes [4, 7, 23], and communication protocols [12]). Node-disjoint paths can be found in the literature for a variety of networks [10–12, 16, 19, 21, 24].

Broadcasting is the problem of disseminating a piece of information, owned by a certain node called the originator, to all other nodes [13]. This is one of the primitives of communication in parallel processing. Hence, inefficient broadcasting can be a bottleneck in the performance of multiprocessor networks. Broadcasting algorithms can be implemented in either a one-port or an all-port model. Under the one-port model, a node can transmit information along at most one incident edge and can simultaneously receive information along at most one incident edge. Under the all-port model, all incident edges of a node can be used simultaneously for information transmission and reception.

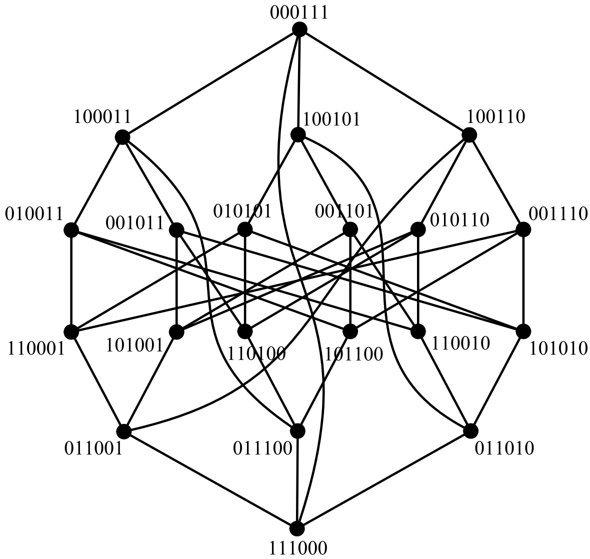
In this paper we construct a maximum number of node-disjoint paths between every two distinct nodes of the folded hyper-star networks. The remainder of this paper is organized as follows: In Sect. 2 we introduce $FHS(2n,n)$ and present some useful properties such as node-symmetry. In Sect. 3 we construct $n+1$ node-disjoint paths in $FHS(2n,n)$ between any two nodes and use them to prove that the fault diameter of $FHS(2n,n)$ is $n+2$ for $n \geq 4$. In Sect. 4 we show a broadcasting scheme of $FHS(2n,n)$. Finally, a conclusion is given in the last section.

2 Preliminaries

The *hyper-star graph* $HS(m,n)$ is an undirected graph consisting of $\binom{m}{n}$ nodes, where each node is represented by a binary string of m bits $b_1 b_2 \cdots b_i \cdots b_m$ such that exactly n bits are 1. Two nodes are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0, or 0 with 1) in another position. The edge connecting two nodes u and v differing in their first and ith bits is called an i-*edge*. Clearly every node in $HS(m,n)$ has degree n or $m-n$, and $HS(m,n)$ is regular if and only if $m=2n$. The *folded hyper-star* $FHS(2n,n)$ is constructed from $HS(2n,n)$ by adding edges to connect pairs of nodes whose binary strings are complement to each other. These edges are called *c-edges*. Let V_n^1 and V_n^0 be the set of nodes that start with 1 and 0 in $FHS(2n,n)$, respectively. Let $\mathrm{dist}(u,v)$ be the distance from $u = u_1 u_2 \cdots u_{2n}$ to $v = v_1 v_2 \cdots v_{2n}$. Let r be the bit string obtained by applying the bitwise Exclusive-OR (denoted by $\oplus$) operation to u and v, so $r = r_1 r_2 \cdots r_{2n}$, where $r_i = u_i \oplus v_i$. It is easy to see that $\mathrm{dist}(u,v) = \min\{\sum_{i=2}^{2n} r_i, 2n - \sum_{i=2}^{2n} r_i\}$.

For $2 \leq i \leq 2n$, let σ_i denote the operation of switching the first and ith bits of a string, and let σ_c denote the operation of switching all bits. For example, $\sigma_3(011100) = 110100$ and $\sigma_c(011100) = 100011$. Note that v-$\sigma_i(v)$ is an edge of $FHS(2n,n)$ if only if the first and ith bits are different in v. For a node u, we will use $[k_1, k_2, \ldots, k_t]$ to denote the path formed by the nodes obtained by applying the operations $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_t}$ successively to u. Clearly every path can be represented in such a way, though not every sequence represents a path. For example, for $u = 000111$, $[4,2,6]$ represents the path 000111–100011–010011–110010, $[5,3,c]$ represents 000111–100101–001101–110010, while $[c,5,3]$ represents 000111–111000–011010–110010 in $FHS(6,3)$. On the other hand, $[4,6,2]$ does not represent a path. Notice that if $k_1, k_2, \ldots, k_t$ are all different integers, and $[k_1, k_2, \ldots, k_t]$ represents a path from u to v, then we can permute the k_i's with even indices, and we can permute the k_i's with odd indices and still get path from u to v of the same length (other permutations do not correspond to paths). Similarly, if c is among the k_i's, then first remove c, permute the remaining numbers in even positions and permute the remaining numbers in odd positions, and then insert c back into the sequence anywhere to get another path from u to v of the same length. The same applies to shortest paths. We will abbreviate the node $\overbrace{0\ldots0}^{n}\overbrace{1\ldots1}^{n}$ in $FHS(2n,n)$ as $0^n 1^n$. Figure 1 shows $FHS(6,3)$.

Fig. 1 *FHS*(6, 3)



The following lemmas describe the cycles in $FHS(2n, n)$.

Lemma 1 *Let u and v be two nodes in $FHS(2n, n)$, and let P and Q be two walks with length $\rho \geq 3$ from u to v represented by $[a_1, \ldots, a_\rho]$ and $[b_1, \ldots, b_\rho]$, respectively. Assume that $a_1, \ldots, a_\rho$ are all different, $b_1, \ldots, b_\rho$ is a permutation of $a_1, \ldots, a_\rho$, neither sequence contains c, and the numbers in odd positions in $a_1, \ldots, a_\rho$ form some cyclically permuted version of the numbers in the odd positions in $b_1, \ldots, b_\rho$. In addition, if ρ is even, assume that $a_\rho \neq b_\rho$. Then P and Q are paths, and connecting them constitutes an even cycle of length 2ρ.*

Proof Since c is not among the numbers representing the paths, and $a_1, \ldots, a_\rho$ are all different, both P and Q are paths, and joining them creates a closed walk. Suppose that there is a common node $w \neq u, v$ on the two paths. Then if the paths from u to w are represented by the subsequences $[a_1, \ldots, a_i]$ and $[b_1, \ldots, b_j]$ for some $i, j < \rho$, and k appears in one of the subsequences but not in the other, then the kth bit of the last node would be different of the two subpaths, so they cannot both end at w. Thus $a_1, \ldots, a_i$ must be a permutation of $b_1, \ldots, b_j$, hence $i = j$. Since the numbers in odd positions in $a_1, \ldots, a_\rho$ form a cyclically permuted version of that of $b_1, \ldots, b_\rho$, the sequence $a_1, \ldots, a_i$ can only be a permutation of $b_1, \ldots, b_i$ if both contain all elements in odd positions. When ρ is odd, this implies $i = \rho$, but then $w = v$. Finally, if ρ is even, we must have $i = \rho - 1$, but then by our assumption $a_\rho \neq b_\rho$, so $a_1, \ldots, a_{\rho-1}$ is not a permutation of $b_1, \ldots, b_{\rho-1}$, a contradiction. Therefore P and Q are internally node-disjoint, and connecting them constitutes an even cycle of length 2ρ. □

Lemma 2 *Let P be a walk in $FHS(2n, n)$ containing exactly one i-edge for each $i = 2, \ldots, 2n$ and exactly one c-edge. Then P is a cycle.*

Proof Let u be an arbitrary node in $FHS(2n, n)$ on P. Since P contains exactly one i-edge for $i = 2, \dots, 2n$ and one c-edge, starting from u each bit is switched twice, so P is closed. Clearly no subpath of P can be closed, so it is a cycle. □

We can easily generalize Lemma 1 for paths containing exactly one c-edge:

Lemma 3 *Let u and v be two nodes in $FHS(2n, n)$, and let P and Q be two walks with length $2 \le \rho \le 2n - 1$ from u to v represented by $[a_1, \dots, a_\rho]$ and $[b_1, \dots, b_\rho]$, respectively. Assume that $a_1, \dots, a_\rho$ are all different, $b_1, \dots, b_\rho$ is a permutation of $a_1, \dots, a_\rho$, $a_\rho \ne b_\rho$, both sequences contain exactly one c, but c is not the first element of both sequences, and when c is deleted from both sequences, the remaining numbers in odd positions in the first sequence form some cyclically permuted version of the remaining numbers in the odd positions in the second sequence. In addition, if ρ is odd, and c appears among the last two elements in both sequences, assume that the last elements of the sequences after c is deleted are different. Then connecting P and Q constitutes an even cycle of length 2ρ.*

Proof Since $a_1, \dots, a_\rho$ are all different, after removing c there are $\rho - 1 \le 2n - 2$ elements left, so at least one integer m such that $2 \le m \le 2n$ is missing from them. Thus for every nonempty subsequence of $a_1, \dots, a_\rho$, if it contains c, then it flips the mth bit exactly once, otherwise every bit corresponding to any element of this subsequence is flipped exactly once, so the corresponding walk cannot be closed. Thus both P and Q are paths, and joining them creates a closed walk. Suppose that there is a common node $w \ne u, v$ on the two paths. Then if the paths from u to w are represented by the subsequences $[a_1, \dots, a_i]$ and $[b_1, \dots, b_j]$ for some $i, j < \rho$, and k appears in one of the subsequences but not in the other, then the kth bit of the last node would be different of the two subpaths, so they cannot both end at w. Thus $a_1, \dots, a_i$ must be a permutation of $b_1, \dots, b_j$, hence $i = j$, and c must appear in both or neither. Since c is not the first element in both sequences, we get $i > 1$, so each subsequence contains the first element different from c of the corresponding sequence. Since the numbers in odd positions in $a_1, \dots, a_\rho$ after deleting c form a cyclically permuted version of that of $b_1, \dots, b_\rho$ after deleting c, the sequence $a_1, \dots, a_i$ can only be a permutation of $b_1, \dots, b_i$ if both contain all elements in odd positions in $a_1, \dots, a_\rho$ after deleting c. Thus there can be at most two elements not contained in the subsequence, c and the last element in an even position when ρ is odd. When ρ is even, this implies $i = \rho - 1$, hence c is the last element of both sequences, contradicting our assumption. Finally, if ρ is odd, we must have $i = \rho - 2$ or $i = \rho - 1$. When $i = \rho - 2$, this implies that the last two elements of the sequences are the same (one of which is c), so after deleting c, the last elements are the same, a contradiction. When $i = \rho - 1$, we get that $a_\rho = b_\rho$, again a contradiction. Therefore P and Q are internally node-disjoint, and connecting them constitutes an even cycle of length 2ρ. □

Note that Lemmas 1 and 3 are not strongest possible; one can easily find a necessary and sufficient condition for two sequences containing the same elements to form a cycle together. However, their statements would be more complicated, and for our purposes, the stated versions suffice.

Theorem 1 *$FHS(2n,n)$ is a bipartite graph that is node-symmetric but not edge-symmetric.*

Proof Clearly nodes starting with 0 form one side of the bipartition, and nodes starting with 1 form the other.

Let $\phi_{i,j}$ denote the operation of exchanging the ith and jth bits of a string for $2 \leq i < j \leq 2n$. It is easy to check that $\phi_{i,j}$ and σ_c are automorphisms of $FHS(2n,n)$ for all $2 \leq i < j \leq 2n$, and it is obvious that using compositions of these automorphisms, we can map any node to any node, thus $FHS(2n,n)$ is node-symmetric.

$FHS(2n,n)$ is clearly not edge-symmetric since any c-edge is part of many 4-cycles of the form $[c,i,c,i]$, but any i-edge is contained in only one 4-cycle for $2 \leq i \leq 2n$. Using the automorphisms above, one can easily show that there are two equivalence classes on the edges. □

3 Node-disjoint paths in $FHS(2n,n)$

It is well known that the hyper-star $HS(2n,n)$ has node-connectivity of n, and in [6] it was shown that adding any perfect matching that only connects nodes whose distance is at least 3 will increase the connectivity by 1. Thus in particular, $FHS(2n,n)$ has node-connectivity of $n+1$. In this section we construct a maximum number of node-disjoint paths in $FHS(2n,n)$ between any two nodes and show that its fault diameter is $n+2$ for $n \geq 4$.

Given a sequence S, we will use $CR_x(S)$ to denote the sequence obtained from S by rotating its elements left x times. For example, if $S = \langle 1,2,\dots,n\rangle$, then $CR_0(S) = \langle 1,2,\dots,n\rangle$ and $CR_3(S) = \langle 4,5,\dots,n,1,2,3\rangle$. Given sequences S_1 and S_2 with $p = q$ or $p = q+1$, $S_1 \otimes S_2$ will represent the new sequence obtained when we alternately pick elements of S_1 and S_2 (finishing with an element of S_1 in case $p = q+1$). For instance, if $S_1 = \langle 5,6,7\rangle$ and $S_2 = \langle 2,3,4\rangle$, then $S_1 \otimes S_2 = \langle 5,2,6,3,7,4\rangle$. The number of different sequences of the form $CR_x(S_1) \otimes CR_x(S_2)$ is $|S_1|$ if S_1 and S_2 has the same number of elements. When S_1 has one more element than S_2, we can get more sequences, since with $k = |S_1|$, the sequence $CR_0(S_1) \otimes CR_0(S_2)$ is different from $CR_k(S_1) \otimes CR_k(S_2)$. For example, if $S_1 = \langle 5,6,7\rangle$ and $S_2 = \langle 2,3\rangle$, then $CR_0(S_1) \otimes CR_0(S_2) = \langle 5,2,6,3,7\rangle$ and $CR_3(S_1) \otimes CR_3(S_2) = \langle 5,3,6,2,7\rangle$. However, these "new" sequences only differ from previous ones in that the numbers in the even positions are cyclically permuted. $c \diamond S_1 \otimes S_2$ will represent the new permutation in which c is added in $S_1 \otimes S_2$ at the beginning. As we have seen before, if c is part of a sequence representing a path in $FHS(2n,n)$, we can move it anywhere in the sequence and still get a path. However, we will only need the ones where c is either in the first or in the last two positions. So let $S_1 \otimes c \diamond S_2$ represent the permutation obtained by adding c to $S_1 \otimes S_2$ in the penultimate position, and let $S_1 \otimes S_2 \diamond c$ represent the permutation obtained by adding c to $S_1 \otimes S_2$ in the last position. For instance, if $S_1 = \langle 5,6,7\rangle$ and $S_2 = \langle 2,3,4\rangle$, then $c \diamond S_1 \otimes S_2 = \langle c,5,2,6,3,7,4\rangle$, $S_1 \otimes S_2 \diamond c = \langle 5,2,6,3,7,4,c\rangle$, and $S_1 \otimes c \diamond S_2 = \langle 5,2,6,3,7,c,4\rangle$. We will use $c \bullet S_1 \otimes S_2$ to denote any or all of $c \diamond S_1 \otimes S_2$, $S_1 \otimes S_2 \diamond c$, and $S_1 \otimes c \diamond S_2$.

Since $FHS(2n,n)$ is node-symmetric, we may fix $u = 0^n 1^n$, and let $v = b_1 b_2 \cdots b_{2n}$ be an arbitrary node of $FHS(2n,n)$. Let the result of applying the bit-wise Exclusive-OR function on these two nodes be the bitstring $r = r_1 r_2 \cdots r_i \cdots r_{2n}$ ($r_i = u_i \oplus b_i$), and let $t = \sum_{i=2}^{2n} r_i$. The set R^1, consisting of bit positions i such that $r_i = 1$ and $2 \le i \le 2n$, is divided into the following two sequences: Bit positions between 2 and n are put in H_1, bit positions that are at least $n+1$ are put into H_2, both in an increasing order. Thus if $R^1 = \{i_1, i_2, \ldots, i_t\}$ such that $2 \le i_1 < i_2 < \cdots < i_g \le n < i_{g+1} < \cdots < i_t$, then $H_1 = \langle i_1, i_2, \ldots, i_g \rangle$ and $H_2 = \langle i_{g+1}, i_{g+2}, \ldots, i_t \rangle$. It is easy to see that $g = \frac{t-1}{2}$ if t is odd, while $g = \frac{t}{2}$ if t is even. The set R^0, consisting of bit positions i such that $r_i = 0$ ($2 \le i \le 2n$) is divided into the two sequences H_3 and H_4 the same way. Thus with $t' = 2n - 1 - t$, if $R^0 = \{i_1, i_2, \ldots, i_{t'}\}$ such that $2 \le i_1 < i_2 < \cdots < i_f \le n < i_{f+1} < \cdots < i_{t'}$, then $H_3 = \langle i_1, i_2, \ldots, i_f \rangle$ and $H_4 = \langle i_{f+1}, i_{f+2}, \ldots, i_{t'} \rangle$. Again it is easy to see that $f = \frac{t'-1}{2}$ if t' is odd, while $f = \frac{t'}{2}$ if t' is even. Hence $|H_2| \ge |H_1|$ and $|H_4| \ge |H_3|$. Using the above notation, we can find paths from u to v of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ and $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$. The lengths of these paths are $|H_1 \cup H_2|$ and $|H_3 \cup H_4| + 1$, respectively, and $\mathrm{dist}(u,v)$ is the smaller of the two. We want to find as many internally node-disjoint paths among these as possible. As we will see later, this will happen as long as the first nodes in the paths are all different and the last nodes in the paths are all different. Among paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$, we can have $|H_2|$ such paths, using $x = 0, \ldots, t-g-1$. Among paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$, we can have $|H_4|$ such paths, using $y = 0, \ldots, t'-f-1$ with c always in the penultimate position. To get one more path, for $y = 0$, we use c in the first and in the last position instead. For example, if $H_3 = \langle 2,3,4 \rangle$ and $H_4 = \langle 5,6,7 \rangle$, the paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ will be $\langle c,5,2,6,3,7,4 \rangle$, $\langle 5,2,6,3,7,4,c \rangle$, $\langle 6,3,7,4,5,c,2 \rangle$, and $\langle 7,4,5,2,6,c,3 \rangle$. And if $H_3 = \langle 2,3 \rangle$ and $H_4 = \langle 5,6,7 \rangle$, the paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ will be $\langle c,5,2,6,3,7 \rangle$, $\langle 5,2,6,3,7,c \rangle$, $\langle 6,3,7,2,c,5 \rangle$, and $\langle 7,2,5,3,c,6 \rangle$. It is easy to see that the first elements of these paths are all different, and the last elements are also different.

In the following lemmas we show that the paths defined this way are internally node-disjoint.

Lemma 4 *The paths of the form* $[CR_x(H_2) \otimes CR_x(H_1)]$ *from* u *to* v *are pairwise internally node-disjoint.*

Proof Fix $0 \le k < l \le t-g-1$. If t is even, then the sequence $CR_l(H_2) \otimes CR_l(H_1)$ is a cyclically permuted version of $CR_k(H_2) \otimes CR_k(H_1)$. If t is odd, then the numbers in the odd positions of the sequence $CR_l(H_2) \otimes CR_l(H_1)$ form a cyclically permuted version of the numbers in the odd positions of $CR_k(H_2) \otimes CR_k(H_1)$. By Lemma 1 in both cases, there is no common node in the two paths. □

Lemma 5 *The paths of the form* $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ *from* u *to* v *are pairwise internally node-disjoint.*

Proof Fix $0 \le k < l \le t'-f-1$. When t' is odd, the numbers in the odd positions in $CR_k(H_4) \otimes CR_k(H_3)$ form a cyclically permuted version of the numbers in the odd positions of $CR_l(H_4) \otimes CR_l(H_3)$. When t' is even, the numbers in $CR_k(H_4) \otimes CR_k(H_3)$ form a cyclically permuted version of the numbers in $CR_l(H_4) \otimes CR_l(H_3)$. By Lemma 3 in both cases, there is no common node in any two such paths represented by the sequences when c is inserted as we described above (first and last position for $y = 0$, penultimate position otherwise). □

Lemma 6 *The paths of the form* $[CR_x(H_2) \otimes CR_x(H_1)]$ *and* $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ *from* u *to* v *are pairwise internally node-disjoint.*

Proof If we pick two paths of the same form, then the claim immediately follows from Lemmas 4 or 5. If the two paths are of different form, then they form a cycle by Lemma 2. Therefore there is no common node in any two of these paths. □

Note that by this lemma we always have $n + 1$ internally node-disjoint paths between any two nodes in $FHS(2n, n)$, so it is maximally connected.

Since $FHS(4, 2)$ is just $K_{3,3}$, if we delete two nodes in it, the remaining graph is either $K_{1,3}$ or $K_{2,2}$, both of which has diameter 2, so its fault diameter is 2. Therefore, we will prove the following about the fault diameter of $FHS(2n, n)$ for $n \geq 3$:

Theorem 2 *The fault diameter of* $FHS(2n, n)$ *is* $n+2$ *when* $n \geq 4$. *The fault diameter of* $FHS(6, 3)$ *is* 4.

Proof We need to find the maximum diameter of $FHS(2n, n)$ after up to n nodes have been deleted. Let two nodes be $u = 0^n 1^n$ and $v = v_1 v_2 \cdots v_{2n}$ in $FHS(2n, n)$. The idea of the proof is as follows: We construct $n + 1$ paths from u to v, each with length at most $n + 2$ in $FHS(2n, n)$, and show that they are node-disjoint. The construction will depend on the distance of v from u. Since $FHS(2n, n)$ is node-symmetric, this will give us the bound of $n + 2$ on the maximum diameter of $FHS(2n, n)$ after the deletion of up to n nodes. Then we show how to delete n nodes to get a diameter of exactly $n + 2$, showing that the fault diameter is $n + 2$.

Define the sequences $H_1, \ldots, H_4$ as before. To construct the paths, we consider the following cases depending on the distance of u and v:

Case 1. $\mathrm{dist}(u, v)$ is odd.

Case 1.1. $\mathrm{dist}(u, v) = n$:

Then n is odd, and we must have $|H_1 \cup H_2| = n$ and $|H_3 \cup H_4| = n - 1$ with $|H_2| = \frac{n+1}{2}$ and $|H_4| = \frac{n-1}{2}$, so we can get $\frac{n+1}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$, and we can get $\frac{n-1}{2} + 1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$. Thus we get $n + 1$ paths in total, and the length of each of these paths is $\mathrm{dist}(u, v) = n$. All these paths are node-disjoint by Lemma 6.

Case 1.2. $\mathrm{dist}(u, v) = n - 1 = |H_1 \cup H_2|$:

Then n is even, and we must have $|H_3 \cup H_4| = n$ with $|H_2| = |H_4| = \frac{n}{2}$, so we can get $\frac{n}{2} + 1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ having length $n + 1$, and we can get $\frac{n}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $n - 1$. Again all these paths are node-disjoint by Lemma 6.

Case 1.3 $\mathrm{dist}(u, v) = n - 1 = |H_3 \cup H_4| + 1$:

Then n is even, and we must have $|H_1 \cup H_2| = n+1$ with $|H_2| = \frac{n+2}{2}$ and $|H_4| = \frac{n-2}{2}$, so we can get $\frac{n+2}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $n+1$ and $\frac{n-2}{2}+1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ having length $n-1$. Again all these paths are node-disjoint by Lemma 6.

Case 1.4. $\mathrm{dist}(u, v) = |H_3 \cup H_4| + 1 \le n-2$:

Then we must have $|H_3| = |H_4| = \frac{\mathrm{dist}(u,v)-1}{2}$ and $|H_1 \cup H_2| = 2n - \mathrm{dist}(u, v)$ with $|H_2| = n - \frac{\mathrm{dist}(u,v)-1}{2}$, so we can get $\frac{\mathrm{dist}(u,v)-1}{2} + 1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ having length $\mathrm{dist}(u, v)$, and we can get $n - \frac{\mathrm{dist}(u,v)-1}{2}$ paths of the form $[\alpha, c \diamond H_3 \otimes H_4, \alpha]$ having length $\mathrm{dist}(u, v) + 2$, where α ranges over the elements of H_2. Since the αth bit of the nodes along the path $[\alpha, c \diamond H_4 \otimes H_3, \alpha]$ is different from the αth bit of the nodes in any of the other paths, these paths are node-disjoint. The length of the longest path among these paths is $\mathrm{dist}(u, v) + 2 \le n < n+2$.

Case 1.5. $\mathrm{dist}(u, v) = |H_1 \cup H_2| \le n-2$:

Then we must have $|H_2| = \frac{\mathrm{dist}(u,v)+1}{2}$ and $|H_3 \cup H_4| = 2n - \mathrm{dist}(u, v) - 1$ with $|H_4| = n - \frac{\mathrm{dist}(u,v)+1}{2}$, so we can get $\frac{\mathrm{dist}(u,v)+1}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $\mathrm{dist}(u, v)$, one path of the form $[c, H_2 \otimes H_1, c]$ having length $\mathrm{dist}(u, v) + 2$ and $n - \frac{\mathrm{dist}(u,v)+1}{2}$ paths of the form $[\beta, \gamma, H_2 \otimes H_1, \beta, \gamma]$ having length $\mathrm{dist}(u, v) + 4$, where β ranges over the elements of H_4, and γ ranges over the elements of H_3 so that each element appears in exactly one such path. (For example, we can take β to be the first element of $CR_y(H_4)$ and γ to be the first element of $CR_y(H_3)$ for $y = 0, 1, \ldots, t' - f - 1$.) Clearly, paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ and $[c, H_2 \otimes H_1, c]$ are node-disjoint. Since both β and γ appear in exactly one of the paths, paths of the form $[\beta, \gamma, H_2 \otimes H_1, \beta, \gamma]$ are node-disjoint, and they are also node-disjoint to paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ and $[c, H_2 \otimes H_1, c]$. Clearly the length of the longest path among these paths is $\mathrm{dist}(u, v) + 4 \le n+2$.

Case 2. $\mathrm{dist}(u, v)$ is even.

Case 2.1. $\mathrm{dist}(u, v) = n$:

Then n is even, and we must have $|H_1 \cup H_2| = n$ and $|H_3 \cup H_4| = n-1$ with $|H_2| = |H_4| = \frac{n}{2}$, so we can get $\frac{n}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ and $\frac{n}{2}+1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$. These paths are node-disjoint by Lemma 6, and their lengths are $\mathrm{dist}(u, v) = n$.

Case 2.2. $\mathrm{dist}(u, v) = |H_3 \cup H_4| + 1 = n-1$:

Then n is odd, and we must have $|H_1 \cup H_2| = n+1$ with $|H_2| = \frac{n+1}{2}$ and $|H_4| = \frac{n-1}{2}$, so we can get $\frac{n-1}{2}+1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ having length $n-1$ and $\frac{n+1}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $n+1$. These paths are again node-disjoint by Lemma 6.

Case 2.3. $\mathrm{dist}(u, v) = |H_1 \cup H_2| = n-1$:

Then n is odd, and we must have $|H_2| = \frac{n-1}{2}$ and $|H_3 \cup H_4| = n$ with $|H_4| = \frac{n+1}{2}$, so we can get $\frac{n-1}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $n-1$ and $\frac{n+1}{2}+1$ paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ having length $n+1$. These paths are again node-disjoint by Lemma 6.

Case 2.4. $\mathrm{dist}(u, v) = |H_3 \cup H_4| + 1 \le n-2$:

Then we must have $|H_4| = \frac{\mathrm{dist}(u,v)}{2}$ and $|H_1 \cup H_2| = 2n - \mathrm{dist}(u, v)$ with $|H_1| = |H_2| = n - \frac{\mathrm{dist}(u,v)}{2}$, so we can get $\frac{\mathrm{dist}(u,v)}{2} + 1$ paths of the form $[c \bullet CR_y(H_4) \otimes$

$CR_y(H_3)]$ having length $\mathrm{dist}(u, v)$ and $n - \frac{\mathrm{dist}(u,v)}{2}$ paths of the form $[\beta, \gamma, c \diamond H_3 \otimes H_4, \beta, \gamma]$ having length $\mathrm{dist}(u, v) + 4$, where β ranges over the elements of H_2, and γ ranges over the elements of H_1 so that each element appears in exactly one such path. Paths of the form $[c \bullet CR_y(H_4) \otimes CR_y(H_3)]$ are node-disjoint by Lemma 5, and since both β and γ appear in exactly one of the paths of the form $[\beta, \gamma, c \diamond H_3 \otimes H_4, \beta, \gamma]$, they are also node-disjoint, and they are also node-disjoint to the other paths. Clearly the length of the longest path among these paths is $\mathrm{dist}(u, v) + 4 \leq n + 2$.

Case 2.5. $\mathrm{dist}(u, v) = |H_1 \cup H_2| \leq n - 2$:

Then we must have $|H_2| = \frac{\mathrm{dist}(u,v)}{2}$ and $|H_3 \cup H_4| = 2n - \mathrm{dist}(u, v) - 1$ with $|H_4| = n - \frac{\mathrm{dist}(u,v)}{2}$, so we can get $\frac{\mathrm{dist}(u,v)}{2}$ paths of the form $[CR_x(H_2) \otimes CR_x(H_1)]$ having length $\mathrm{dist}(u, v)$, one path of the form $[c, H_2 \otimes H_1, c]$ having length $\mathrm{dist}(u, v) + 2$, and $n - \frac{\mathrm{dist}(u,v)}{2}$ paths of the form $[\alpha, H_1 \otimes H_2, \alpha]$ having length $\mathrm{dist}(u, v) + 2$, where α ranges over the elements of H_4. Since each α appears in exactly one path, these paths are all node-disjoint, and the length of the longest path among them is $\mathrm{dist}(u, v) + 2 \leq n < n + 2$.

Now we know that the fault diameter of $FHS(2n, n)$ is at most $n + 2$. To show that it is $n + 2$, we need to specify which n nodes to delete to get the diameter to become $n + 2$. Let $n \geq 4$. First assume that n is odd, so $n \geq 5$. Pick a node v such that $\mathrm{dist}(u, v) = n - 2 \geq 3$ corresponding to Case 1.5. Delete nodes $\sigma_i(u)$ and $\sigma_i(v)$ for all $i \in H_2$ and node $\sigma_c(v)$, altogether $2|H_2| + 1 = \mathrm{dist}(u, v) + 2 = n$ nodes, to get graph G (note that u and v have not been deleted). Now for any path from u to v in G, the last edge must be a j-edge for some $j \in H_3$ since all other edges incident to v have been removed. Since $|H_3 \cup H_4| = n + 1$, all shortest paths from u to $\sigma_j(v)$ in $FHS(2n, n)$ (having length $n - 1$) are of the form $[S_1 \otimes S_2]$, where S_1 is a permutation of H_2, and S_2 is a permutation of $H_1 \cup \{j\}$, so they must all start with an i-edge for some $i \in H_2$. Since all these edges have been removed, and $FHS(2n, n)$ is bipartite, the shortest path from u to $\sigma_j(v)$ has length at least $n + 1$ in G, so the distance from u to v in G is at least $n + 2$. When n is even, the argument is similar: pick a node v such that $\mathrm{dist}(u, v) = n - 2 \geq 2$ corresponding to Case 2.4, and delete all nodes $\sigma_i(u)$ and $\sigma_i(v)$ for all $i \in H_4$ and nodes $\sigma_c(u)$ and $\sigma_c(v)$, altogether $2|H_4| + 2 = n$ nodes, to get graph G. Then for any path from u to v in G, the last edge must be a j-edge for some $j \in H_1$, and all shortest paths from u to $\sigma_j(v)$ are obtained by inserting c into $[S_1 \otimes S_2]$, where S_1 is a permutation of H_4, and S_2 is a permutation of $H_3 \cup \{j\}$. Thus every such shortest path must start with an i-edge for some $i \in H_4$, but none of them are available in G.

Hence the fault diameter of $FHS(2n, n)$ is $n + 2$ when $n \geq 4$. For $n = 3$, the above example fails, since if $\mathrm{dist}(u, v) = n - 2 = 1$, then u and v are neighbors, so their distance will still be 1 after deleting any nodes other than u and v. From the other cases we can see that the fault diameter of $FHS(6, 3)$ is at most $n + 1 = 4$, and deleting node 100011 in $FHS(6, 3)$ will give a graph of diameter 4, so the fault diameter of $FHS(6, 3)$ is 4. □

4 One-to-all broadcasting of $FHS(2n, n)$

In [14], the one-to-all broadcasting algorithm of $HS(2n, n)$ under the one-port model was introduced. We can find a one-to-all broadcasting algorithm of $FHS(2n, n)$ under the one-port model using the algorithm of $HS(2n,n)$. Now we propose the one-to-all broadcasting algorithm of $HS(2n,n)$ and $FHS(2n,n)$ under the all-port model. A layered graph consists of nodes in $l+1$ layers, which are numbered L_0 to L_l, such that each node is in one of the layers, and each edge connects nodes in consecutive layers. Many multiprocessor networks can be represented as layered graphs, for example, the hypercube. $HS(2n,n)$ and $FHS(2n,n)$ can also be represented as layered graphs as follows: A node u is chosen to be the only node in layer L_0, and for every other node v, if $\mathrm{dist}(u,v)=k$, then node v is put in layer L_k. Then $HS(2n,n)$ has layers from L_0 to L_{2n-1}, and every edge connects nodes in consecutive layers. And $FHS(2n,n)$ has layers from L_0 to L_n, and every edge connects nodes in consecutive layers (e.g., to get the representation of $FHS(6,3)$ from Fig. 1, the bottom vertex 111000 is put to layer L_1, while the three vertices above it are put to layer L_2). Using the concept of the layered graph, we mention the one-to-all broadcasting algorithm of under $HS(2n,n)$ all-port model:

Let node u in L_k hold the message M, $k=0$. Then all of the nodes in L_k send M to all nodes in L_{k+1}, then set $k:=k+1$. This operation is performed repeatedly until $k+1=2n-1$.

This scheme takes $2n-1$ time, which is optimal, since the diameter of $HS(2n,n)$ is $2n-1$ [17]. We can find that the one-to-all broadcasting algorithm of $FHS(2n,n)$ under all-port model similar to the algorithm of $HS(2n,n)$. The one-to-all broadcasting time of $FHS(2n,n)$ under all-port model is n, which is optimal, since the diameter of $FHS(2n,n)$ is n [17].

5 Conclusion

The folded hyper-star networks $FHS(2n,n)$ were introduced to be a competitive model to both the hypercubes and the star graphs. Some basic parameters of $FHS(2n,n)$ were provided including size, degree, diameter, shortest path routing scheme, etc. In this paper, we analyzed some more good properties of $FHS(2n,n)$. We proved that $FHS(2n,n)$ is node-symmetric, constructed maximum number of node-disjoint paths in $FHS(2n,n)$ between any two nodes, and used that to show that the fault diameter of $FHS(2n,n)$ is $n+2$ for $n\geq 4$. Also, we suggested a one-to-all broadcasting algorithm of $HS(2n,n)$ and $FHS(2n,n)$ under the all-port model.

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